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Lecture 4

Frequency Domain Analysis and Fourier Transform

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In an earlier lecture, I talked about the idea of "complex frequency" s,

where $s = \sigma + j\omega$.

Using such a concept of "complex frequency" allows us to analyse signals and systems with better generality.

In this lecture, we will restrict ourselves to just consider the theoretical foundation of **frequency domain analysis** where we assume $\sigma = 0$. In particular, we will examine the mathematics related to **Fourier Transform**, which is one of the most important aspects of signal processing.

Relationship between exponentials and sinusoids
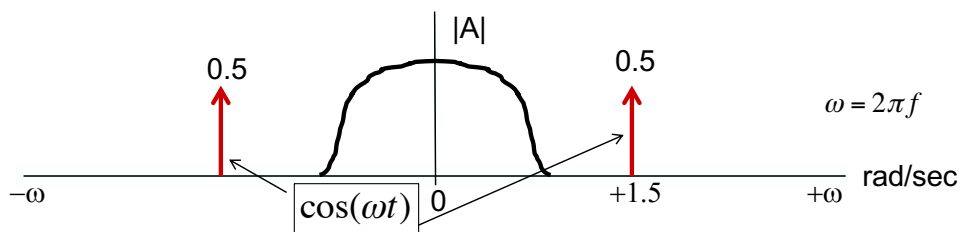
◆ Euler's formula:

$$e^{j\omega t} = \cos(\omega t) + j\sin(\omega t)$$

$$e^{-j\omega t} = \cos(-\omega t) + j\sin(-\omega t)$$
$$= \cos(\omega t) - j\sin(\omega t)$$

$$\cos(\omega t) = \frac{e^{j\omega t} + e^{-j\omega t}}{2}$$

$$\sin(\omega t) = \frac{e^{j\omega t} - e^{-j\omega t}}{2j}$$

◆ Therefore, in signal analysis, we usual regard "frequency" to be ω in the exponential vector:

◆ The frequency spectrum is therefore a plot of the amplitude (and phase) projected onto exponential components $e^{j\omega t}$ for different ω.

Before we consider Fourier Transform, it is important to understand the relationship between **sinusoidal signals** and **exponential functions**. So far we have been using sine and cosine functions because they **are physically realisable** and easy to understand. However, in signal processing, we often use the exponentials $e^{j\omega t}$ to represent the "**bases functions**", i.e. the underlying building blocks to a signal.

Euler's equations provide the link between $e^{j\omega t}$ and the sine/cosine functions.

What is the physical interpretation of the exponential term: $e^{j\omega t}$? We can regard this as a VECTOR with unit magnitude, rotating in a complex plane at a rate of ω rads/sec in the direction shown. Unity vector because the magnitude of the vector $|e^{j\omega t}|$ is 1.

Now cos(ωt) and sin(ωt) are just the projection of the this vector on the REAL (x-axis) and IMAGINERY (y-axis) axes in this diagram.

There are two important implications of this view of the spectrum:

1. There are both +ve and –ve frequency components in signals. In fact we need both together in order to make physical signals such as the cosine or sine waves.

2. The +ve & -ve components are symmetrical (or antisymmetrical) for real signals.
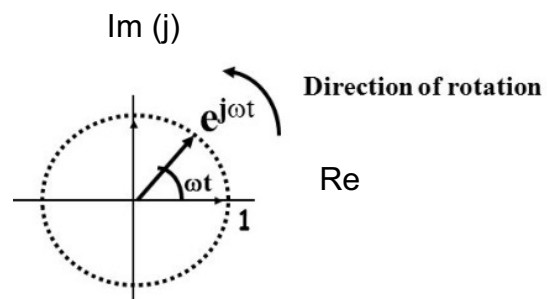
Fourier Series in three forms
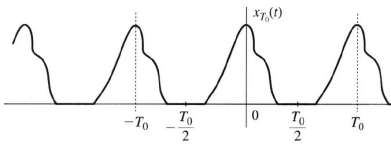
$$x(t) = a_0 + \sum_{n=1}^{\infty}(a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)$$

$$a_n = \frac{2}{T_0}\int_0^{T_o} x(t)\,\cos n\omega_0 t\, dt \qquad b_n = \frac{2}{T_0}\int_0^{T_o} x(t)\,\sin n\omega_0 t\, dt$$

$$x(t) = C_0 + \sum_{n=1}^{\infty} C_n \cos(n\omega_0 t + \theta_n)$$

$$C_n = \sqrt{a_n^2 + b_n^2}$$
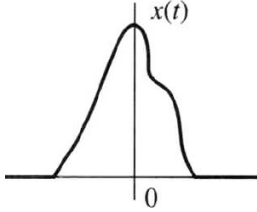
$$\theta_n = tan^{-1}\left(\frac{b_n}{a_n}\right)$$

$$x(t) = \sum_{-\infty}^{\infty} D_n\, e^{j(n\omega_0 t + \theta_n)} \qquad D_n = \frac{1}{T_0}\int_{-T_{o/2}}^{T_{o/2}} x(t)e^{-jn\omega_0 t}dt$$

Everlasting exponentials provides us with a third form of expressing the Fourier series representation of a periodic signal as shown in this slide.
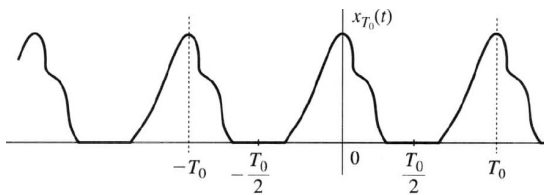
Definition of Fourier Transform

◆ The forward and inverse **Fourier Transform** are defined for **aperiodic** signal as:

$$X(\omega) = \mathcal{F}[x(t)] \quad = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

$$x(t) = \mathcal{F}^{-1}[X(\omega)] \quad = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega$$

◆ **Fourier series** is used for **periodic** signals.

$$x(t) = \sum_{-\infty}^{\infty} D_n e^{j(n\omega_0 t + \theta_n)}$$

$$D_n = \frac{1}{T_0}\int_{-T_{0/2}}^{T_{0/2}} x(t)e^{-jn\omega_0 t}dt$$

Here is the formal definition of the **Fourier Transform**. It is important to note that the Fourier Transform as defined in this equation here is **applicable only to aperiodic signals**. This is because the limits of the integral are from $-\infty$ to $+\infty$.

You can also interpret the term: $X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$ as computing the among of stuff (i.e. projection) in *x(t)* that can be found in the building block $e^{j\omega t}$.

As can be seen in the inverse Fourier Transform equation, x(t) is made up of adding together (the integral) the weighted sum of $e^{j\omega t}$ components at all different frequencies ω. The weighting for each frequency component at ω is X(ω).

The $^1/_{2\pi}$ scaling is to account for the relationship between frequency in Hz and in rads/sec (f = $\omega/2\pi$).

Now you are more familiar with Fourier Series than Fourier Transform. So what's the difference? Fourier Transform is actually more "physically real" because any real-world signal MUST have finite energy, and must therefore be aperiodic. Fourier Series is applicable only to periodic signals, which has infinite signal energy. However, it turns out that Fourier series is most useful when using computers to process signals. As we will see in a later lecturer, Discrete Fourier Transform is based on Fourier Series.

The main difference between the two is that for Fourier Series, since the signal is periodic, frequency components are discrete and are INTEGRAL MULTIPLE of a base frequency known as the **fundamental frequency** fo. fo is 1/To, where To is the periodicity of the signal (i.e. period of repetition).
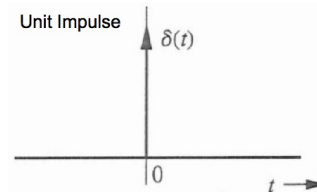
Define three useful functions

♦ A unit rectangular window function **rect(x)**:

$$\text{rect}(x) = \begin{cases} 0 & |x| > \frac{1}{2} \\ \frac{1}{2} & |x| = \frac{1}{2} \\ 1 & |x| < \frac{1}{2} \end{cases}$$

♦ The unit impulse function **δ(t)** (Dirac impulse):

$$\delta(t) = 0 \qquad t \neq 0$$

$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1$$

♦ Interpolation function **sinc(x):**

$$\text{sinc}(x) = \frac{\sin x}{x} \qquad \text{or} \qquad \text{sinc}(x) = \frac{\sin \pi x}{\pi x}$$

Here are three "theoretic" functions or waveforms that pop up often in signal processing. Although they are theoretical, they allow us to model real-world signals in a way that helps understanding.

Firstly is the **rectangular function,** which we often call this a "**window**" because when we multiple it with a signal, it lets some signals through, and blocks others, just like a window letting in a specific view.

Second is something we already considered in Lecture 1, the **unit impulse** function (or Dirac Function). As we will see, this has an important role in signal processing and signal modeling, particularly as we SAMPLE a continuous time signal to get a discrete time signal.

Finally, there is this interesting $\mathbf{sinc(x)} = \frac{\sin x}{x}$ function, which we will consider in some details in subsequent slides.

More about sinc(x) function

- **sinc(x)** is an even function of x.

- **sinc(x)** = 0 when sin(x) = 0 except when x=0, i.e. x = ±π, ±2π, ±3π…..

- **sinc(0)** = 1 (derived with L'Hôpital's rule)

- **sinc(x)** is the product of an oscillating signal sin(x) and a monotonically decreasing function 1/x. Therefore it is a damping oscillation with period of 2π with amplitude decreasing as 1/x.
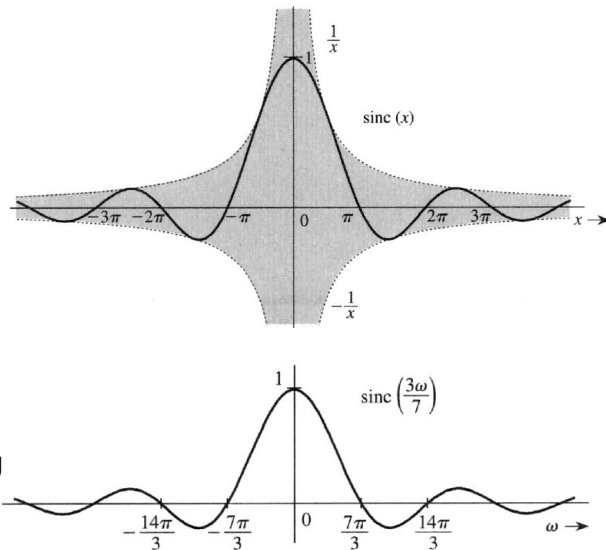
The sinc-function is important in signals. It can be view as an oscillatory signal sin(x) with its amplitude monotonically decreasing as time goes to ±infinity.
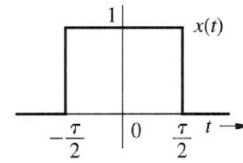
At time zero, sinc(0) = 1 (can be proved, but we will not do so here). As x → ±∞ sinc(x) → 0.

The period of oscillation is governed by the sin(x) term.
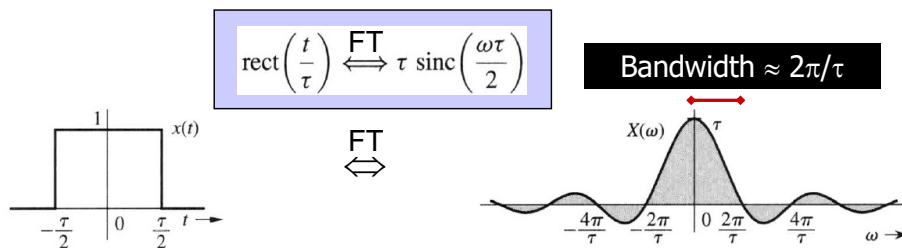
Fourier Transform of $x(t) = \text{rect}(t/\tau)$

- ◆ Evaluation:

$$X(\omega) = \int_{-\infty}^{\infty} \text{rect}\left(\frac{t}{\tau}\right) e^{-j\omega t} dt$$

- ◆ Since rect(t/τ) = 1 for -τ/2 < t < τ/2 and 0 otherwise

$$X(\omega) = \int_{-\tau/2}^{\tau/2} e^{-j\omega t} dt \quad = -\frac{1}{j\omega}(e^{-j\omega\tau/2} - e^{j\omega\tau/2}) = \frac{2\sin\left(\frac{\omega\tau}{2}\right)}{\omega} \quad = \tau \frac{\sin\left(\frac{\omega\tau}{2}\right)}{\left(\frac{\omega\tau}{2}\right)} = \tau \,\text{sinc}\left(\frac{\omega\tau}{2}\right)$$

$$\text{rect}\left(\frac{t}{\tau}\right) \overset{FT}{\Longleftrightarrow} \tau\,\text{sinc}\left(\frac{\omega\tau}{2}\right)$$

Bandwidth ≈ 2π/τ

$$\overset{FT}{\Longleftrightarrow}$$

The reason that sinc-function is important is because the **Fourier Transform of a rectangular window rect(t/τ) is a sinc-function**.

This is interesting because if we extract a section of a signal to analyse, and obtain its spectrum (via Fourier Transform), we are effectively multiplying the signal with a rectangular function (rect()). Therefore we can expect that the original signal's frequency spectrum will, in some way, be modified by this multiplication process.

It turns out that indeed the frequency spectrum of the original signal is changed according to the sinc function – the spectral representation of the rectangular window! The way such "modification" is done is through a process known as "**convolution**", something we will study a bit later.
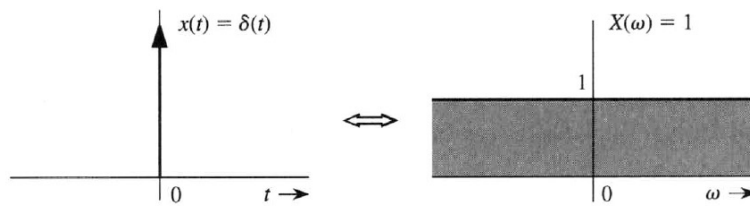
Fourier Transform of unit impulse $x(t) = \delta(t)$

◆ Using the sampling property of the impulse, we get:

$$\mathcal{F}[\delta(t)] = \int_{-\infty}^{\infty} \delta(t)e^{-j\omega t}dt = 1$$

◆ IMPORTANT – Unit impulse contains COMPONENT AT EVERY FREQUENCY.

$$\delta(t) \Longleftrightarrow 1$$

Let us return to the unit impulse function $\delta(t)$ or the delta function. The FT of $\delta(t)$ is very simple: a constant 1. What does this mean? It means that to make up $\delta(t)$, we need infinite number of equal frequency components!

Or another way of looking at it, an impulse contains ALL frequencies. Of course this is only a mathematical model. We cannot have a real-world signal with ALL frequency components. However, we can approximate an impulse by having a very narrow pulse, as compared to the window width of the signal.
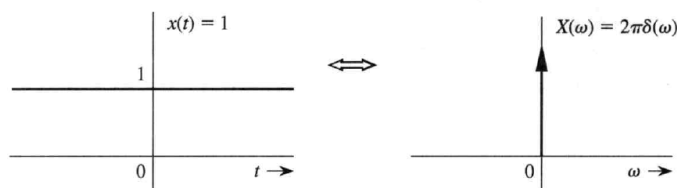
Inverse Fourier Transform of $\delta(\omega)$

♦ Using the sampling property of the impulse, we get:

$$\mathcal{F}^{-1}[\delta(\omega)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} \delta(\omega)e^{j\omega t}d\omega = \frac{1}{2\pi}$$

♦ Spectrum of a constant (i.e. d.c.) signal x(t) = 1 is an impulse $2\pi\delta(\omega)$.

$$\frac{1}{2\pi} \iff \delta(\omega) \qquad \text{or} \qquad 1 \iff 2\pi\,\delta(\omega)$$

What is the **inverse Fourier Transform of an impulse** in the frequency spectrum?

The integral term is zero everywhere, except when $\omega = 0$, then $\delta(\omega) = 1$. But when $\omega = 0$, $e^{j\omega t} = 1$. Therefore the integral term (which is really area under the curve) is simply = 1.

Therefore, if the impulse is at zero frequency, (at $\omega = 0$), the time domain equivalent signal is simply a DC voltage (i.e. a constant). You are familiar with this from the First Year – DC signal is simply one that has zero frequency!

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Inverse Fourier Transform of $\delta(\omega - \omega_0)$

◆ Using the sampling property of the impulse, we get:

$$\mathcal{F}^{-1}[\delta(\omega - \omega_0)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} \delta(\omega - \omega_0)e^{j\omega t}\, d\omega = \frac{1}{2\pi}e^{j\omega_0 t}$$

◆ Spectrum of an everlasting exponential $e^{j\omega_0 t}$ is a single impulse at $\omega = \omega_0$.

$$\frac{1}{2\pi}e^{j\omega_0 t} \iff \delta(\omega - \omega_0)$$

or

$$e^{j\omega_0 t} \iff 2\pi\delta(\omega - \omega_0)$$

and

$$e^{-j\omega_0 t} \iff 2\pi\delta(\omega + \omega_0)$$

If the impulse is at a non-zero frequency (at $\omega = \omega_0$) in the frequency domain (i.e. an impulse in the spectrum), we have an **everlasting exponential** $e^{j\omega t}$ at $\omega = \omega_0$ in the time domain.

In other words, the Fourier Transform of an everlasting exponential $e^{j\omega_0 t}$ is an impulse in the frequency spectrum at $\omega = \omega_0$.
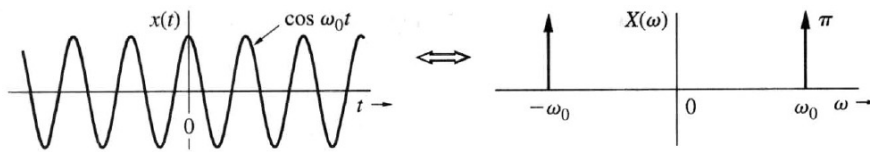
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- Remember Euler's formula: $\cos \omega_0 t = \frac{1}{2}(e^{j\omega_0 t} + e^{-j\omega_0 t})$

- Use results from previous slide, we get:

$$\cos \omega_0 t \iff \pi[\delta(\omega + \omega_0) + \delta(\omega - \omega_0)]$$

- Spectrum of cosine signal has two impulses at positive and negative frequencies.

An everlasting exponential $e^{j\omega t}$ is a mathematical model. It has both real part and imaginary part (see Slide 2).

In order to get a "real-world" signal such as an everlasting cosine wave, we need to have **both the positive and negative frequency components**. Using Euler formula, we get:

$$\cos \omega_0 t = \frac{1}{2}(e^{j\omega_0 t} + e^{-j\omega_0 t})$$

So, from the previous slide, we can conclude that the spectrum of an everlasting cosine signal has two impulses at ω_0 and $-\omega_0$, and the magnitude is π.

Fourier Transform of any periodic signal

◆ Fourier series of a periodic signal x(t) with period T_0 is given by:

$$x(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t} \qquad \omega_0 = \frac{2\pi}{T_0}$$

◆ Take Fourier transform of both sides, we get:

$$X(\omega) = 2\pi \sum_{n=-\infty}^{\infty} D_n \delta(\omega - n\omega_0)$$

◆ This is rather obvious!

What about the Fourier Transform of a periodic signal? The easiest way to think about this is to first represent the periodic signal in Fourier Series form. That is, the periodic signal is a sum (not integral) of everlasting exponentials at discrete frequencies, where the frequencies are integral multiple of the fundamental frequency ω_0. (ω_0 is determined by the periodicity of the signal).

Based on what we know from the previous slide, that the FT of an everlasting exponential is an impulse at the corresponding frequency, we can conclude that the FT of a periodic signal (i.e. its spectrum) consists of weighted impulses at $n\omega_o$, where is all +ve and –ve integers, and the weight is D_n, the Fourier coefficient for the nth harmonic.
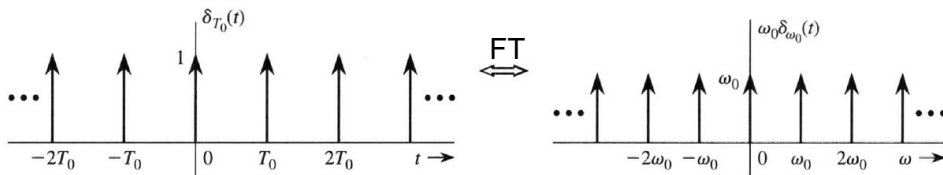
Fourier Transform of a unit impulse train

- Consider an impulse train $\delta_{T_0}(t) = \sum_{-\infty}^{\infty} \delta(t - nT_0)$

- The Fourier series of this impulse train can be shown to be:

$$\delta_{T_0}(t) = \sum_{-\infty}^{\infty} D_n e^{jn\omega_0 t} \quad \text{where} \quad \omega_0 = \frac{2\pi}{T_0} \quad \text{and} \quad D_n = \frac{1}{T_0}$$

- Therefore using results from the last slide (slide 12), we get:

$$X(\omega) = \frac{2\pi}{T_0} \sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_0) \qquad \omega_0 = \frac{2\pi}{T_0}$$

$$= \omega_0 \delta_{\omega_0}(\omega)$$

Finally, let us consider the FT of an impulse train – i.e. a periodic sequence of impulses, at interval T_0. We can show that the Fourier series of an impulse train is simply sum of everlasting exponentials at the fundamental frequency and its harmonics, and the weighting Dn is the constant 1/To:

$$\delta_{T_0}(t) = \sum_{-\infty}^{\infty} D_n e^{jn\omega_0 t} \quad \text{where} \quad \omega_0 = \frac{2\pi}{T_0} \quad \text{and} \quad D_n = \frac{1}{T_0}$$

Using our results from the previous slide, we can conclude that the FT of a period impulse train in the time domain is a series of impulses in the frequency domain – a beautiful result!

Fourier Transform Table (1)

No.	$x(t)$	$X(\omega)$	
1	$e^{-at}u(t)$	$\dfrac{1}{a + j\omega}$	$a > 0$
2	$e^{at}u(-t)$	$\dfrac{1}{a - j\omega}$	$a > 0$
3	$e^{-a|t|}$	$\dfrac{2a}{a^2 + \omega^2}$	$a > 0$
4	$te^{-at}u(t)$	$\dfrac{1}{(a + j\omega)^2}$	$a > 0$
5	$t^n e^{-at}u(t)$	$\dfrac{n!}{(a + j\omega)^{n+1}}$	$a > 0$
6	$\delta(t)$	1	
7	1	$2\pi\delta(\omega)$	
8	$e^{j\omega_0 t}$	$2\pi\delta(\omega - \omega_0)$	

I have provided here and in the next two slides a table of signals (functions) and their Fourier Transform in a closed-form. DO NOT MEMBERISE ANY OF THESE. They are provided here as a reference – something for you to look up in the future.

In my exams, I will NOT require you to remember any of these. I may, however, ask you to derive some simple cases from FIRST PRINCIPLE.

Fourier Transform Table (2)

No.	$x(t)$	$X(\omega)$	
9	$\cos \omega_0 t$	$\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$	
10	$\sin \omega_0 t$	$j\pi[\delta(\omega + \omega_0) - \delta(\omega - \omega_0)]$	
11	$u(t)$	$\pi\delta(\omega) + \dfrac{1}{j\omega}$	
12	$\operatorname{sgn} t$	$\dfrac{2}{j\omega}$	
13	$\cos \omega_0 t\, u(t)$	$\dfrac{\pi}{2}[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)] + \dfrac{j\omega}{\omega_0^2 - \omega^2}$	
14	$\sin \omega_0 t\, u(t)$	$\dfrac{\pi}{2j}[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)] + \dfrac{\omega_0}{\omega_0^2 - \omega^2}$	
15	$e^{-at} \sin \omega_0 t\, u(t)$	$\dfrac{\omega_0}{(a + j\omega)^2 + \omega_0^2}$	$a > 0$
16	$e^{-at} \cos \omega_0 t\, u(t)$	$\dfrac{a + j\omega}{(a + j\omega)^2 + \omega_0^2}$	$a > 0$

Fourier Transform Table (3)

No.	$x(t)$	$X(\omega)$	
16	$e^{-at}\cos\omega_0 t\, u(t)$	$\dfrac{a+j\omega}{(a+j\omega)^2+\omega_0^2}$	$a>0$
17	$\operatorname{rect}\left(\dfrac{t}{\tau}\right)$	$\tau\operatorname{sinc}\left(\dfrac{\omega\tau}{2}\right)$	
18	$\dfrac{W}{\pi}\operatorname{sinc}(Wt)$	$\operatorname{rect}\left(\dfrac{\omega}{2W}\right)$	
19	$\Delta\left(\dfrac{t}{\tau}\right)$	$\dfrac{\tau}{2}\operatorname{sinc}^2\left(\dfrac{\omega\tau}{4}\right)$	
20	$\dfrac{W}{2\pi}\operatorname{sinc}^2\left(\dfrac{Wt}{2}\right)$	$\Delta\left(\dfrac{\omega}{2W}\right)$	
21	$\displaystyle\sum_{n=-\infty}^{\infty}\delta(t-nT)$	$\displaystyle\omega_0\sum_{n=-\infty}^{\infty}\delta(\omega-n\omega_0)$	$\omega_0=\dfrac{2\pi}{T}$
22	$e^{-t^2/2\sigma^2}$	$\sigma\sqrt{2\pi}\,e^{-\sigma^2\omega^2/2}$	

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Three Big Ideas

1. **Euler formula** provides an alternative way to represent sine and cosine functions in terms of $e^{j\omega t}$ and $e^{-j\omega t}$.

$$\cos(\omega t) = \frac{e^{j\omega t} + e^{-j\omega t}}{2} \qquad \sin(\omega t) = \frac{e^{j\omega t} - e^{-j\omega t}}{2j}$$

2. **Extracting a portion of a signal** x(t) for $-\tau/2 \le t \le \tau/2$ can be modelled by multiplying x(t) by the rectangular function rect(x/τ).

3. The Fourier Transform of an infinite train of unit impulses is again an infinite train of unit impulses.

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